$\boxed{\#9}$ 9(a) $f(x)=e^x,\ f(0)=1$

$f'(x)=e^x,\ f'(0)=1$

$f''(x)=e^x,\ f''(0)=1$

$\vdots$

$f^{(n)}(x)=e^x,\ f^{(n)}(0)=1$

$\Rightarrow \therefore f(x)=e^x=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(x-0)^n=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ ▨

9(b) $\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right|=\lim_{n\to\infty}\frac{|x|}{n+1}=0<1\ \forall x\in(-\infty,\infty)$

$\therefore$ the interval of conv. $=(-\infty,\infty)$ ▨

(9c) 在背面

MA1004: CALCULUS II, Spring 2006

FINAL EXAM: Chapter 8 ~ Chapter 10

June 20, 2006

$\boxed{\#1}$. $f'(x)=\frac{-\csc x^2\cot x^2(2x)-(-\csc^2 x^2)(2x)}{\csc x^2-\cot x^2}=\frac{2x(\csc x^2)\cancel{(\csc x^2-\cot x^2)}}{\cancel{\csc x^2-\cot x^2}}=2x\cdot\csc x^2$

1. (10%) Find the derivative of the function $f(x)=\ln|\csc x^2-\cot x^2|$.

2. (10%) Use implicit differentiation to find $\frac{dy}{dx}$ and evaluate the derivative at the given point: $\sin x+\cos 2y=1$, $(x,y)=(\pi/2,\pi/4)$.

$\boxed{\#2}$ $\sin x+\cos 2y=1$

$\Rightarrow \frac{d}{dx}(\sin x+\cos 2y)=\frac{d}{dx}1=0$

$\Rightarrow \cos x-\sin 2y\cdot 2\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=\frac{\cos x}{2\sin 2y}$

$(x,y)=(\frac{\pi}{2},\frac{\pi}{4})\Rightarrow \frac{dy}{dx}=\frac{\cos\frac{\pi}{2}}{2\sin\frac{\pi}{2}}=\frac{0}{2}=0$

3. (10%) Evaluate the indefinite integral: $\int e^{\sin x}\cos x\,dx$.

$\boxed{\#3}$ let $u=\sin x,\ \frac{du}{dx}=\cos x$ $\therefore \int e^{\sin x}\cos x\,dx=\int e^u\cos x\cdot\frac{1}{\cos x}du=\int e^u du=e^u+C=e^{\sin x}+C$

4. (10%) Evaluate $\lim_{x\to-\infty}\frac{x^2}{e^{-x}}$ and $\lim_{x\to 1}\frac{2\ln x}{e^x}$.

$\boxed{\#4}$

(i) $\lim_{x\to-\infty}\frac{x^2}{e^{-x}}\overset{L}{=}\lim_{x\to-\infty}\frac{2x}{-e^{-x}}\overset{L}{=}\lim_{x\to-\infty}\frac{2}{e^{-x}}=0$

(ii) $\lim_{x\to 1}\frac{2\ln x}{e^x}=\frac{2\ln 1}{e^1}=\frac{2\cdot 0}{e}=0$

5. (10%) Consider the infinite geometric series: $\sum_{n=0}^{\infty}ar^n$, where $a\neq 0$. Prove that if $|r|<1$, then the geometric series converges to $\frac{a}{1-r}$.

$\boxed{\#5}$ $S_0=a,\ S_1=a+ar,\ S_2=a+ar+ar^2$ $\cdots\ S_n=a+ar+\cdots+ar^n \Rightarrow rS_n=ar+ar^2+\cdots+ar^{n+1}$

$\Rightarrow S_n-rS_n=a-ar^{n+1}$

$\Rightarrow S_n=\frac{a(1-r^{n+1})}{1-r},\ r\neq 1$

$\lim_{n\to\infty}S_n=\frac{a}{1-r}$ ▨ $\left(\because |r|<1,\ \lim_{n\to\infty}r^{n+1}=0\right)$

6. (10%) Determine the convergence or divergence of the series $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$ by the ratio test.

$\boxed{\#6}$ $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^2}\cdot\frac{n^2}{2^n}\right|=\lim_{n\to\infty}2\left(\frac{n}{n+1}\right)^2=2>1$ $\therefore$ divergent!!

7. (10%) Determine the convergence or divergence of the series $\sum_{n=1}^{\infty}\frac{n!}{2(n!)+1}$.

$\boxed{\#7}$ $\because \lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n!}{2(n!)+1}=\lim_{n\to\infty}\frac{1}{2+\frac{1}{n!}}=\frac{1}{2}\neq 0$ $\therefore$ divergent!!

8. Suppose that $f(x)=\frac{1}{x}$ can be represented by a power series centered at $c=1$.

   (a) (10%) Find the power series.

   (b) (10%) Find the interval of convergence of the power series.

9. Suppose that $f(x)=e^x$ can be represented by a power series centered at $c=0$.

   (a) (10%) Find the power series.

   (b) (10%) Find the interval of convergence of the power series.

   (c) (10%) Use the eighth-degree Taylor polynomial for e^{-x^2} to approximate the definite integral $\int_0^1 e^{-x^2}dx$.

$\boxed{\#8}$ 8(a)

$f(x)=\frac{1}{x}=x^{-1},\ f(1)=1$

$f'(x)=-x^{-2},\ f'(1)=-1=-1!$

$f''(x)=(-1)(-2)x^{-3},\ f''(1)=(-1)(-2)=2!$

$f'''(x)=(-1)(-2)(-3)x^{-4},\ f'''(1)=-3!$

$\vdots$

$\therefore f(x)=\frac{1}{x}=f(1)+\frac{f'(1)}{1!}(x-1)^1+\frac{f''(1)}{2!}(x-1)^2+\cdots$

$=1+(-1)(x-1)+\frac{2!}{2!}(x-1)^2+\frac{-3!}{3!}(x-1)^3+\cdots$

$=1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-\cdots$

$=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$ ▨

8(b) $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-1)^{n+1}}{(-1)^n(x-1)^n}\right|=\lim_{n\to\infty}|x-1|=|x-1|$

$\therefore$ the series conv. if $|x-1|<1 \Leftrightarrow x\in(0,2)$

$x=0$: $\sum_{n=0}^{\infty}(-1)^n(x-1)^n=\sum_{n=0}^{\infty}(-1)^n(-1)^n=\sum_{n=0}^{\infty}1$

$\therefore$ div ($\because \lim_{n\to\infty}1=1\neq 0$)

$x=2$: $\sum_{n=0}^{\infty}(-1)^n(x-1)^n=\sum_{n=0}^{\infty}(-1)^n 1^n=\sum_{n=0}^{\infty}(-1)^n$

$\therefore$ div ($\because \lim_{n\to\infty}(-1)^n\neq 0$)

$\therefore$ the interval of convergence $=(0,2)$ ▨

9(c): By 9(a), $e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots -$ $\forall x \in (-\infty, \infty)$

$\therefore S_8(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}$

$$\therefore \int_0^1 e^{-x^2}dx \cong \int_0^1 S_8(x)dx = \int_0^1 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}\,dx$$

$$= \left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2} - \frac{x^7}{7\cdot 6} + \frac{x^9}{9\cdot 24}\right)\Big]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216}$$

$$\cong 0.747$$ ///