# MA1004: CALCULUS II, Spring 2006

## MIDTERM 2: Section 6.6 ~ Chapter 7

May 16, 2006

1. (10%) Evaluate $\int_{-\infty}^{\infty} 2xe^{-3x^2}dx$.

   **Solution:**

$$\begin{aligned}\int_{-\infty}^{\infty} 2xe^{-3x^2}dx &= \int_{-\infty}^{0} 2xe^{-3x^2}dx + \int_{0}^{\infty} 2xe^{-3x^2}dx \\ &= \lim_{a\to-\infty}\left[-\frac{1}{3}e^{-3x^2}\right]_a^0 + \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3x^2}\right]_0^b \\ &= (-\frac{1}{3}+0)+(0+\frac{1}{3}) \\ &= 0.\end{aligned}$$

2. (10%) Evaluate $\int_{0}^{2} \frac{1}{(x-1)^{1/3}}dx$.

   **Solution:**

$$\begin{aligned}\int_{0}^{2} \frac{1}{(x-1)^{1/3}}dx &= \int_{0}^{1} \frac{1}{(x-1)^{1/3}}dx + \int_{1}^{2} \frac{1}{(x-1)^{1/3}}dx \\ &= \lim_{a\to1^-}\left[\frac{3}{2}(x-1)^{2/3}\right]_0^a + \lim_{b\to1^+}\left[\frac{3}{2}(x-1)^{2/3}\right]_b^2 \\ &= -\frac{3}{2}+\frac{3}{2} \\ &= 0.\end{aligned}$$

3. (10%) Apply the second-partials test to find the relative extrema and saddle points of $f(x,y) = \frac{1}{3}(24xy - 6x^2y - 4xy^2)$.

   **Solution:**

$$\begin{aligned}f_x(x,y) &= \frac{1}{3}y(24-12x-4y), \\ f_y(x,y) &= \frac{1}{3}x(24-6x-8y), \\ f_{xx}(x,y) &= -4y, \\ f_{yy}(x,y) &= -\frac{8}{3}x, \\ f_{xy}(x,y) &= 8-4x-\frac{8}{3}y.\end{aligned}$$

Let $f_x(x,y) = 0$ and $f_y(x,y) = 0$. We find the critical points are $(0,0)$, $(0,6)$, $(4,0)$, and $(\frac{4}{3}, 2)$.

(a) $(0,0)$: $d = f_{xx}(0,0)f_{yy}(0,0) - (f_{xy}(0,0))^2 = -64 < 0$, $\therefore (0,0,f(0,0))$ is a saddle point.

(b) $(0,6)$: $d = f_{xx}(0,6)f_{yy}(0,6) - (f_{xy}(0,6))^2 = -64 < 0$, $\therefore (0,6,f(0,6))$ is a saddle point.

(c) $(4,0)$: $d = f_{xx}(4,0)f_{yy}(4,0) - (f_{xy}(4,0))^2 = -64 < 0$, $\therefore (4,0,f(4,0))$ is a saddle point.

(d) $(4/3,2)$: $d = f_{xx}(4/3,2)f_{yy}(4/3,2) - (f_{xy}(4/3,2))^2 = 192/9 > 0$ and $f_{xx}(4/3,2) = -8 < 0$, $\therefore f(4/3,2) = 64/9$ is a relative maximum.

4. (10%) Use the method of Lagrange multipliers to find the maximum value of $f(x,y,z) = xyz$ subject to the constraints $x^2 + z^2 = 5$ and $x - 2y = 0$ (Assume that $x, y, z \geq 0$).

**Solution:**

Let $F(x,y,z,\lambda,\mu) := xyz - \lambda(x^2 + z^2 - 5) - \mu(x - 2y)$. Then

$$\begin{aligned}
F_x &= yz - 2x\lambda - \mu = 0, \\
F_y &= xz + 2\mu = 0 \Rightarrow \mu = -\frac{xz}{2}, \\
F_z &= xy - 2z\lambda = 0 \Rightarrow \lambda = \frac{xy}{2z}, \\
F_\lambda &= -(x^2 + z^2 - 5) = 0 \Rightarrow z = \sqrt{5 - x^2}, \\
F_\mu &= -(x - 2y) = 0 \Rightarrow y = \frac{x}{2}.
\end{aligned}$$

From $F_x$, we have

$$\frac{x\sqrt{5-x^2}}{2} - \frac{x^3}{2\sqrt{5-x^2}} + \frac{x\sqrt{5-x^2}}{2} = 0.$$

Then

$$\begin{aligned}
x\sqrt{5-x^2} &= \frac{x^3}{2\sqrt{5-x^2}}, \\
2x(5-x^2) &= x^3, \\
3x^3 - 10x &= 0, \\
x(3x^2 - 10) &= 0.
\end{aligned}$$

Since $x, y, z \geq 0$, we have $x = \sqrt{10/3}$, $y = \frac{1}{2}\sqrt{10/3}$, $z = \sqrt{5/3}$.

$\therefore$ The maximum value is $f(\sqrt{10/3}, \frac{1}{2}\sqrt{10/3}, \sqrt{5/3}) = \frac{5\sqrt{15}}{9}$.

5. (20%) The production function for a company is given by $f(x, y) = 100x^{0.25}y^{0.75}$, where $x$ is the number of units of labor and $y$ is the number of units of capital. Suppose that labor costs \$48 per unit, capital costs \$36 per unit, and management sets a production goal of 20000 units. Find the numbers of units of labor and capital needed to meet the production goal while minimizing the cost.

**Solution:** Let $F(x, y, \lambda) := 48x + 36y - \lambda(x^{0.25}y^{0.75} - 200)$. Then

$$\begin{aligned} F_x &= 48 - 0.25\lambda x^{-0.75}y^{0.75} = 0, \\ F_y &= 36 - 0.75\lambda x^{0.25}y^{-0.25} = 0, \\ F_\lambda &= -(x^{0.25}y^{0.75} - 200) = 0. \end{aligned}$$

$\therefore (\frac{y}{x})^{0.75} = \frac{48}{0.25\lambda}$ and $(\frac{y}{x})^{0.25} = \frac{0.75\lambda}{36}$

$\therefore \frac{y}{x} = (\frac{48}{0.25\lambda})(\frac{0.75\lambda}{36}) = 4$

$\therefore x = \frac{200}{4^{0.75}} = \frac{200}{2\sqrt{2}} = 50\sqrt{2}$

$y = 4x = 200\sqrt{2}$

6. (10%) Find the values of $a$ and $b$ such that the linear model $f(x) = ax + b$ has a minimum sum of the squared errors for the given points: $(-2, -1), (0, 0), (2, 3)$.

**Solution:**

$$\begin{aligned} S &= (-2a + b + 1)^2 + (0a + b)^2 + (2a + b - 3)^2, \\ \frac{\partial S}{\partial a} &= 2(-2a + b + 1)(-2) + 2(2a + b - 3)(2) = 16a - 16, \\ \frac{\partial S}{\partial b} &= 2(-2a + b + 1) + 2b + 2(2a + b - 3) = 6b - 4. \end{aligned}$$

Let $\frac{\partial S}{\partial a} = 0$ and $\frac{\partial S}{\partial b} = 0$. We have $a = 1$ and $b = \frac{2}{3}$.

7. (10%) Evaluate the following integral over the region $R$: $\int\int_R \frac{y}{x^2 + y^2} dA$, where $R$ is the triangle bounded by $y = x$, $y = 2x$, and $x = 2$.

**Solution:**

$$\begin{aligned} \int\int_R \frac{y}{x^2 + y^2} dA &= \int_0^2 \int_x^{2x} \frac{y}{x^2 + y^2} dy dx = \int_0^2 \left[\frac{1}{2}\ln(x^2 + y^2)\right]_x^{2x} dx \\ &= \frac{1}{2}\int_0^2 [\ln(5x^2) - \ln(2x^2)] dx \\ &= \frac{1}{2}\int_0^2 \ln(\frac{5}{2}) dx = (\frac{1}{2}\ln(\frac{5}{2}))x\Big]_0^2 = \ln(\frac{5}{2}). \end{aligned}$$

8. (20%) Evaluate the following integral: $\int_0^1 \int_{y/2}^{1/2} e^{-x^2} dxdy$.

**Solution:**

$$\begin{aligned}
\int_0^1 \int_{y/2}^{1/2} e^{-x^2} dxdy &= \int_0^{1/2} \int_0^{2x} e^{-x^2} dydx = \int_0^{1/2} 2xe^{-x^2} dx \\
&= -e^{-x^2}\Big]_0^{1/2} = 1 - e^{-1/4}.
\end{aligned}$$

9. (10%) A company sells two products whose demand functions are given by

$$x_1 = 500 - 3p_1 \quad \text{and} \quad x_2 = 750 - 2.4p_2.$$

So, the total revenue is given by

$$R = x_1p_1 + x_2p_2.$$

Estimate the average revenue if the price $p_1$ varies between \$50 and \$75, and the price $p_2$ varies between \$100 and \$150.

**Solution:**

$$\begin{aligned}
\text{average} &= \frac{1}{25 \times 50} \int_{100}^{150} \int_{50}^{75} [(500 - 3p_1)p_1 + (750 - 2.4p_2)p_2] dp_1 dp_2 \\
&= \frac{1}{1250} \int_{100}^{150} \int_{50}^{75} [-3p_1^2 + 500p_1 - 2.4p_2^2 + 750p_2] dp_1 dp_2 \\
&= \frac{1}{1250} \int_{100}^{150} \left[ -p_1^3 + 250p_1^2 - 2.4p_1p_2^2 + 750p_1p_2 \right]_{50}^{75} dp_2 \\
&= \frac{1}{1250} \int_{100}^{150} (484375 - 60p_2^2 + 18750p_2) dp_2 \\
&= \frac{1}{1250} \left[ 484375p_2 - 20p_2^3 + 9375p_2^2 \right]_{100}^{150} \\
&= 75125.
\end{aligned}$$