MA 1004: CALCULUS II, Spring 2006

MIDTERM 1: Chapter 5 $\sim$ §6.5

March 28, 2006

1. (10%) The cost per unit c of producing a product over a two-year period is modeled by

$$c = 0.005t^2 + 0.01t + 13.15, \qquad 0 \leq t \leq 24,$$

where t is the time in months. Approximate the average cost per unit over the two-year period.

Solution:

$$\begin{aligned} \text{average cost per unit} &= \frac{1}{24}\int_0^{24}(0.005t^2 + 0.01t + 13.15)dt \\ &= \cdots \\ &= 14.23. \end{aligned}$$

2. (10%) The demand and supply functions for a product are modeled by

$$\text{demand} : p_1(x) = \frac{10000}{\sqrt{x+100}}, \qquad \text{supply} : p_2(x) = 100\sqrt{0.05x+10}.$$

Find the consumer and producer surpluses.

Solution:

$$\begin{aligned} \because \quad & \frac{10000}{\sqrt{x+100}} = 100\sqrt{0.05x+10} \Longrightarrow x = 300 \text{ and } p = 500 \\ \therefore \quad & \text{CS} = \int_0^{300}(\frac{10000}{\sqrt{x+100}} - 500)dx = \cdots = 50000 \\ & \text{PS} = \int_0^{300}(500 - 100\sqrt{0.05x+10})dx = \frac{10000}{3}(4\sqrt{10} - 5) \cong 25497 \end{aligned}$$

3. (10%) Find the area of the region bounded by the graphs of the functions $f(x) = e^{0.5x}$ and $g(x) = -1/x$ for $1 \leq x \leq 2$.

Solution:

$$A = \int_1^2 (e^{0.5x} - (-\frac{1}{x}))dx = \cdots = (2e + \ln 2) - 2e^{0.5}.$$

4. (10%) Find the volume of the solid formed by revolving the region bounded by the graphs of the equations $y = x^2$ and $y = 4x - x^2$ about x-axis.

Solution:

$$\because \quad x^2 = 4x - x^2 \Longrightarrow x = 0, 2$$
$$\therefore \quad V = \pi \int_0^2 (4x - x^2)^2 - (x^2)^2 dx = \cdots = \frac{32}{3}\pi$$

5. (10%) Use the error formula to find n such that the error in the approximation of the definite integral $\int_1^3 e^{2x}\,dx$ is less than 0.0001 using Simpson's rule. Note that the error E in approximating $\int_a^b f(x)\,dx$ is

$$|E| \leq \frac{(b-a)^5}{180n^4} \max_{a \leq x \leq b} |f^{(4)}(x)|,$$

and $e^6 \simeq 403.4287934927351$.

Solution:

$$\begin{aligned}
& f(x) = e^{2x}, f'(x) = 2e^{2x}, f''(x) = 4e^{2x}, f^{(3)}(x) = 8e^{2x}, f^{(4)}(x) = 16e^{2x} \\
\Longrightarrow \quad & \max_{[1,3]} |f^{(4)}(x)| = f^{(4)}(3) = 16e^6 \cong 6454.861. \\
\Longrightarrow \quad & |E| \leq \frac{(3-1)^5}{180n^4}(6454.861) < 0.0001 \\
\Longrightarrow \quad & n^4 > 11475308.44 \\
\Longrightarrow \quad & n > 58.2 \\
\therefore \quad & n \geq 60.
\end{aligned}$$

6. (64%) Find the indefinite and definite integrals:

(6a) $\displaystyle\int \frac{1}{x\sqrt{x+1}}\,dx$ (6b) $\displaystyle\int \frac{1}{e^x+1}\,dx$

(6c) $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$ (6d) $\displaystyle\int x^2(\ln x)\,dx$

(6e) $\displaystyle\int x(\ln x)^2\,dx$ (6f) $\displaystyle\int_0^4 (2 - |x-2|)\,dx$

(6g) $\displaystyle\int_0^1 x^2 e^x\,dx$ (6h) $\displaystyle\int_0^1 x e^{x^2}\,dx$

Solution:

(6a) Hint: Let $u = \sqrt{x+1}$. Then by substitution $\Rightarrow \int \frac{1}{x\sqrt{x+1}}\,dx = 2\int \frac{1}{u^2-1}du$. By partial fractions $\Rightarrow 2\int \frac{1}{u^2-1}du = \ln|u-1| - \ln|u+1| + C = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$.

(6b) Hint: Let $u = e^x$. Then by substitution $\Rightarrow \int \frac{1}{e^x+1}\,dx = \int \frac{1}{u(u+1)}du$. By partial fractions $\Rightarrow \int \frac{1}{u(u+1)}du = \ln e^x - \ln(e^x+1) + C = x - \ln(e^x+1) + C$.

(6c) Hint: By partial fractions $\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx = \int \frac{6}{x} + \frac{-1}{x+1} + \frac{9}{(x+1)^2}dx = 6\ln|x| - \ln|x+1| - \frac{9}{x+1} + C = \ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + C$.

(6d) Hint: By integration by parts, $\int x^2 \ln x dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{1}{x}dx = \cdots = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$.

(6e) Hint: By integration by parts twice, $\int x(\ln x)^2 dx = \frac{1}{2}x^2(\ln x)^2 - \int x\ln x dx = \cdots = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C$.

(6f) Hint: $\int_0^4 (2-|x-2|)\,dx = \int_0^2 2+(x-2)dx + \int_2^4 2-(x-2)dx = \cdots = 4$.

(6g) Hint: Integration by parts twice, we have $\int x^2e^x dx = x^2e^x - 2\int xe^x dx = \cdots = x^2e^x - 2xe^x + 2e^x + C$.
$\therefore \int_0^1 x^2e^x dx = (x^2e^x - 2xe^x + 2e^x)\Big]_0^1 = e-2$.

(6h) Hint: Let $u = x^2$. Then by substitution,

$$\int_0^1 xe^{x^2}dx = \int_0^{1^2} xe^u\frac{1}{2x}du = \int_0^1 \frac{1}{2}e^u du = \frac{1}{2}(e-1).$$