MA 1003: CALCULUS I, Fall 2005
Final Exam: Chapter 4 $\sim$ Section 5.3

January 10, 2006

1. (10%) Suppose that P dollars is deposited in an account at an annual interest rate of r. What is the balance A after t years if compounded n times per year? What is the balance A after t years if compounded continuously?

   Solution:

   (a) $A = P(1 + \frac{r}{n})^{nt}$.

   (b) $A = \lim_{n\to\infty} P(1 + \frac{r}{n})^{nt} = \lim_{n\to\infty} P\Big((1 + \frac{r}{n})^{\frac{n}{r}}\Big)^{rt} = Pe^{rt}$.

2. (10%) Let $4x^3 + \ln y^2 + 2y = 2x$. Find $\frac{dy}{dx}$.

   Solution:

   $$\frac{d}{dx}\Big(4x^3 + \ln y^2 + 2y\Big) = \frac{d}{dx}\Big(2x\Big)$$
   $$\Leftrightarrow \quad 12x^2 + 2\frac{1}{y}\frac{dy}{dx} + 2\frac{dy}{dx} = 2$$
   $$\Leftrightarrow \quad \frac{dy}{dx} = \frac{2 - 12x^2}{\frac{2}{y} + 2} = \frac{y(1 - 6x^2)}{y + 1}$$

3. (10%) Let $f(x) = \ln(x^2\sqrt{x^2 + 1})$. Find $f'(x)$.

   Solution:

   $f(x) = \ln(x^2\sqrt{x^2 + 1}) = \ln x^2 + \ln(x^2 + 1)^{1/2} = 2\ln x + \frac{1}{2}\ln(x^2 + 1)$.

   $f'(x) = \frac{2}{x} + \frac{1}{2}\frac{2x}{x^2 + 1} = \frac{2}{x} + \frac{x}{x^2 + 1}$.

4. (10%) Let $g(x) = \ln\Big(\frac{e^x + e^{-x}}{2}\Big)^{1/3}$. Find $g'(x)$.

   Solution:

   $g(x) = \ln\Big(\frac{e^x + e^{-x}}{2}\Big)^{1/3} = \frac{1}{3}\ln(e^x + e^{-x}) - \ln 2$.

   $g'(x) = \frac{1}{3}\Big(\frac{e^x - e^{-x}}{e^x + e^{-x}}\Big)$.

5. (10%) Let $f(x) = 5^x$. Find the second derivative of $f(x)$.

   Solution:

   $f(x) = 5^x = e^{x\ln 5}$.

   $f'(x) = e^{x\ln 5}\ln 5$.

   $f''(x) = e^{x\ln 5}(\ln 5)^2 = (\ln 5)^2 5^x$.

6. (10%) Find the points of inflection of the function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

   Solution:

   $f'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}(-x)$.

$f''(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}(x^2 - 1), \quad \forall\, x \in \mathbb{R}.$

Let $f''(x) = 0 \Rightarrow x = \pm 1.$

$\therefore$ The test interval are $(-\infty - 1)$, $(-1, 1)$, and $(1, \infty)$

(a) $(-\infty, -1): \ f''(-2) > 0, \quad \therefore f$ is CU on $(-\infty, -1)$

(b) $(-1, 1): \ f''(0) < 0, \quad \therefore f$ is CD on $(-1, 1)$

(c) $(1, \infty): \ f''(2) > 0, \quad \therefore f$ is CU on $(1, \infty)$

By (a)(b)(c), we know that the points of inflection are $(1, f(1))$ and $(-1, f(-1))$.

7. (10%) Find the revenue and demand functions for the given marginal revenue: $\frac{dR}{dx} = 225 + 2x - x^2$. (Hint: use the fact that $R = 0$ when $x = 0$)

Solution:

$R(x) = \int (225 + 2x - x^2)\ dx = 225x + x^2 - \frac{x^3}{3} + C$

$\because\ R = 0$ when $x = 0$

$\therefore C = 0$

$\therefore R(x) = \int (225 + 2x - x^2)\ dx = 225x + x^2 - \frac{x^3}{3}$

$\because R(x) = p(x)x$

$\therefore p(x) = 225 + x - \frac{x^2}{3}$ is the demand function

8. (10%) Find the equation of the function f whose graph passes through the point $(0, 4/3)$ and whose derivative is $f'(x) = x\sqrt{1 - x^2}$.

Solution:

$f(x) = \int x\sqrt{1 - x^2}dx = -\frac{1}{2}\int (1 - x^2)^{1/2}(-2x)dx = -\frac{1}{2}(\frac{2}{3})(1 - x^2)^{3/2} + C$

$\because f(0) = \frac{4}{3} \qquad \therefore C = \frac{5}{3}$

$\therefore f(x) = -\frac{1}{3}(1 - x^2)^{3/2} + \frac{5}{3}$

9. (10%) Find the indefinite integral: $\int \frac{1}{x \ln x} dx.$

Solution:

$\int \frac{1}{x \ln x} dx = \int \frac{1}{\ln x}\frac{1}{x} dx = \ln|\ln x| + C$

10. (10%) Find the indefinite integral: $\int \frac{1 + e^{-x}}{1 + xe^{-x}} dx.$

Solution:

$\int \frac{1 + e^{-x}}{1 + xe^{-x}} dx = \int \frac{e^x + 1}{e^x + x} dx = \ln|e^x + x| + C$