MA 1003: CALCULUS I, Fall 2005

Midterm 3: Chapter 3

December 13, 2005

In problems $1 \sim 4$, $f(x) = 2x^3 - 3x^2 - 36x + 14$.

1. (10%) Find the critical numbers of f.

   Solution:

   $f'(x) = 6x^2 - 6x - 36$

   Let $f'(x) = 0$. Then $x = 3$ or $x = -2$

   $\therefore$ the critical numbers of f are $-2,\ 3$

2. (10%) Find the open intervals on which the function f is increasing or decreasing, and find all relative extrema of the function f.

   Solution:

   The test intervals are $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$.

   ....

   ....

   $\therefore f(-2) = 58$ is a relative maximum $\quad f(3) = -67$ is a relative minimum

3. (10%) Discuss the concavity of the graph of f.

   Solution:

   $f''(x) = 12x - 6$

   Let $f''(x) = 0$. Then $x = 1/2$.

   $\therefore$ the test intervals are $(-\infty, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$

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   f is CD on $(-\infty, \frac{1}{2})$ and CU on $(\frac{1}{2}, \infty)$.

4. (10%) Find the points of inflection of the graph of f.

   Solution:

   By problem 3, $(\frac{1}{2}, f(\frac{1}{2}))$ is a point of inflection.

In problems $5 \sim 7$, $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$. $D_f = \{x \in \mathbf{R} | x \neq \pm 2\}$.

5. (10%) Find the vertical and horizontal asymptotes of the graph of f.

   Solution:

   Let $x^2 - 4 = 0$. Then $x = 2$ or $x = -2$.

$\because 2(x^2-9) \neq 0$ for $x=2$ or $x=-2$

$\therefore x=2$ and $x=-2$ are two vertical asymptotes of f

Moreover, $\lim_{x\to 2^+} f(x) = -\infty$, $\lim_{x\to 2^-} f(x) = \infty$, $\lim_{x\to -2^+} f(x) = \infty$, $\lim_{x\to -2^-} f(x) = -\infty$.

$\because \lim_{x\to\infty} f(x) = \cdots\cdots = 2$

and $\lim_{x\to -\infty} f(x) = \cdots\cdots = 2$

$\therefore y=2$ is a horizontal asymptote of f

6. (10%) Find the critical number and relative extrema of f.

   Solution:

   $f'(x) = \cdots = \dfrac{20x}{(x^2-4)^2}$

   Let $f'(x)=0$. Then $x=0$.

   $\therefore$ the critical number of f is $x=0$

   $\because -2, 2 \notin D_f$

   $\therefore$ the test intervals are $(-\infty,-2)$, $(-2,0)$, $(0,2)$, and $(2,\infty)$

   ....

   ....

   $f(0) = 9/2$ is a relative minimum.

7. (10%) Discuss the concavity of the graph of f.

   Solution:

   $f''(x) = \cdots = \dfrac{-20(3x^2+4)}{(x^2-4)^3}$

   $\because f''(x) \neq 0, \quad \forall\, x \in D_f = \mathbb{R} \setminus \{\pm 2\}$

   $\therefore$ the test intervals are $(-\infty,-2)$, $(-2,2)$, and $(2,\infty)$

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   f is CD on $(-\infty,-2)$, CU on $(-2,2)$, and CD on $(2,\infty)$

8. (10%) Find the absolute extrema of the function $h(t) = (t-1)^{\frac{2}{3}}$ on the closed interval $[-7,2]$.

   Solution:

   $h'(t) = \dfrac{2}{3}(t-1)^{-1/3} = \dfrac{2}{3(t-1)^{1/3}}, \quad \forall\, t\ in(-7,2) \setminus \{1\}$

   $\because h$ is not diff. at $t=1$

   $\therefore$ the only critical number of h on $(-7,2)$ is $t=1$

$h(1) = 0$: abs. min.

$h(-7) = 4$: abs. max.

$h(2) = 1$

9. (10%) Find the points on the graph of $y = 4 - x^2$ that are closest to the point $(0, 2)$.

   Solution: See textbook, page 204, Example 3.

10. (10%) A commodity has a demand function modeled by $p = 100 - 0.5x^2$ and a total cost function modeled by $C = 40x + 37.5$. (1). What price yields a maximum profit? (2). When the profit is maximized, what is the average cost per unit?

    Solution:

    (1)

$$\begin{aligned} P &= xp - C \\ &= x(100 - \frac{1}{2}x^2) - (40x + 37.5) \\ &= -\frac{1}{2}x^3 + 60x - 37.5 \end{aligned}$$

$\therefore P'(x) = -\frac{3}{2}x^2 + 60$

Let $P'(x) = 0$. Then $x = \sqrt{40} = 2\sqrt{10}$.

One can check that P is increasing on $(0, \sqrt{40})$ and decreasing on $(\sqrt{40}, \infty)$.

$\therefore$ The max. profit $P(x)$ occurs at $x = \sqrt{40}$ and the price $p = 100 - 40/2 = 80$.

(2) $\overline{C} = 40 + \frac{37.5}{x}$

$\therefore \overline{C}(\sqrt{40}) = 40 + \frac{37.5}{\sqrt{40}}$ is the average cost per unit