MA 1003: CALCULUS I, Fall 2005

Midterm 2: Chapter 2 $\sim$ Section 3.1

November 15, 2005

1. (15%) You decide to form a partnership with another business. Your business determines that the demand x for your product is inversely proportional to the square of the price p for $x \geq 5$.

   (a) Assume that if the price is \$1000 then the demand is 16 units. Find the demand function.

   (b) Your partner determines that the product costs \$250 per unit and the fixed cost is \$10000. Find the cost function.

   (c) Find the marginal profit for $x = 10$.

   Solution: $x = \dfrac{c}{p^2}$ for $x \geq 5 \Rightarrow p = \sqrt{\dfrac{c}{x}}$ for $x \geq 5$.

   (a) $1000 = \sqrt{\dfrac{c}{16}} = \dfrac{\sqrt{c}}{4}$.
   $\Rightarrow \sqrt{c} = 4000$.
   $\Rightarrow$ the demand function is $p = \dfrac{4000}{\sqrt{x}}$ for $x \geq 5$.

   (b) $C(x) = 10000 + 250x$.

   (c) $R(x) = px = \dfrac{4000}{\sqrt{x}}x = 4000\sqrt{x}$.
   $\Rightarrow P(x) = R(x) - C(x) = 4000\sqrt{x} - 250x - 10000$.
   $\Rightarrow \dfrac{dP}{dx} = \dfrac{2000}{\sqrt{x}} - 250$.
   $\Rightarrow \dfrac{dP}{dx}(10) = \dfrac{2000}{\sqrt{10}} - 250$.

2. (15%) Consider the function $f(x) = \dfrac{x^4 + 1}{x^2}$.

   (a) Find the domain of f.

   (b) Find the critical numbers of f.

   (c) Find the intervals on which the function f is increasing or decreasing.

   Solution:

   (a) $D_f = \{x \in \mathbb{R} |\ x \neq 0\} = \mathbb{R} \setminus \{0\}$.

   (b) $f'(x) = \cdots = \dfrac{2(x^4 - 1)}{x^3}$.
   Let $f'(x) = 0$. Then $x = \pm 1$ are the critical numbers.

(c) $\because x = \pm 1$ are the critical numbers and $x = 0$ is a discontinuity of $f(x)$
$\therefore$ the test intervals are $(-\infty, -1]$, $[-1, 0)$, $(0, 1]$, and $[1, \infty)$
$x \in (-\infty, -1]$: $\because f'(-2) < 0 \quad \therefore f$ is decreasing on $(-\infty, -1]$
$x \in [-1, 0)$: $\because f'(\frac{-1}{2}) > 0 \quad \therefore f$ is increasing on $[-1, 0)$
$x \in (0, 1]$: $\because f'(\frac{1}{2}) < 0 \quad \therefore f$ is decreasing on $(0, 1]$
$x \in [1, \infty)$: $\because f'(2) > 0 \quad \therefore f$ is increasing on $[1, \infty)$

3. (20%) Determine whether the following statements are true or false. Give reasons to justify your conclusions.

(a) If $f'(x) = g'(x)$, then $f(x) = g(x)$.
(b) If $y = f(x)g(x)$, then $y' = f'(x)g'(x)$.
(c) If $f'(c) = g'(c) = 0$ and $h(x) = f(x)g(x)$, then $h'(c) = 0$.
(d) If $f(x)$ is an nth-degree polynomial, then $f^{(n+1)}(x) = 0$.

Solution:

(a) False! Counterexample: $f(x) = x + 1$, $g(x) = x \Rightarrow f(x) \neq g(x)$ but $f'(x) = 1 = g'(x)$.
(b) False! Counterexample: $f(x) = x^2$, $g(x) = x$, $y = x^3 \Rightarrow \dfrac{dy}{dx} = 3x^2 \neq 2x = f'(x)g'(x)$.
(c) True! $\because h'(c) = f'(c)g(c) + f(c)g'(c) = 0$.
(d) True! Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$.
Then $\cdots$, $f^{(n)}(x) = a_n n!$, and $f^{(n+1)}(x) = 0$.

4. (10%) Use the limit definition to find the derivative of the function $f(x) = \dfrac{1}{x+2}$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x+2} - \frac{1}{x+2}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{x+2-(x+\Delta x+2)}{(x+\Delta x+2)(x+2)}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x+2)(x+2)} \\ &= \frac{-1}{(x+2)^2}. \end{aligned}$$

5. (10%) Find the tangent line of the function $f(x) = \left(\dfrac{x+5}{x-1}\right)(2x+1)$ at the point $(0, -5)$.

Solution:

$\because f'(x) = \cdots = \frac{-6(2x+1)}{(x-1)^2} + 2\frac{x+5}{x-1}$.

$\therefore f'(0) = -16$.

$\therefore$ the tangent line is $y - (-5) = -16(x - 0)$. i.e., $y = -16x - 5$.

6. (10%) $y = \left(\dfrac{6-5x}{x^2-1}\right)^2$. Find $\dfrac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = 2\left(\frac{6-5x}{x^2-1}\right)\frac{(-5)(x^2-1)-(6-5x)(2x)}{(x^2-1)^2} = \frac{2(6-5x)(5x^2-12x+5)}{(x^2-1)^3}.$$

7. (10%) Find the second derivative of $f(x) = \dfrac{x}{x^2+3}$.

Solution:

$$\begin{aligned} f'(x) &= \frac{-x^2+3}{(x^2+3)^2}, \\ f''(x) &= \cdots = \frac{2x(x^2-9)}{(x^2+3)^3}. \end{aligned}$$

8. (10%) Find the tangent line to the graph of $x^3 + 5xy^3 = 6xy$ at the point $(1, 1)$.

Solution:

$\because x^3 + 5xy^3 = 6xy$

$\therefore \frac{d}{dx}(x^3 + 5xy^3) = \frac{d}{dx}(6xy)$

$\therefore 3x^2 + 5y^3 + 5x3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$

$\therefore \dfrac{dy}{dx} = \dfrac{3x^2 + 5y^3 - 6y}{6x - 15xy^2}$.

$\therefore \dfrac{dy}{dx}|_{(x,y)=(1,1)} = -\dfrac{2}{9}$

$\therefore$ the tangent line is $y - 1 = -\frac{2}{9}(x - 1)$