MA 1003: CALCULUS I, Fall 2005
Midterm 1: Chapter 0 $\sim$ Chapter 1
October 18, 2005

1. (10%) Find the solution set of the inequality: $x^2 < x + 6$.

   Solution:

$$\begin{aligned} \because\quad & x^2 < x + 6 \\ \Leftrightarrow\quad & x^2 - x - 6 < 0 \\ \Leftrightarrow\quad & (x-3)(x+2) < 0 \end{aligned}$$

   $\therefore$ the solution set is $\{x \in \mathbb{R} : -2 < x < 3\}$, i.e., $(-2, 3)$

2. (10%) Use the Rational Zero Theorem as an aid in finding all real zeros of the polynomial: $6x^3 - 11x^2 - 19x - 6$.

   Solution:

   Factors of constant term: ± 1, ± 2, ± 3, ± 6

   Factors of leading coefficient: ± 1, ± 2, ± 3, ± 6

   By the Rational Zero Theorem, the possible rational zeros are ± 1, ± 2, ± 3, ± 6, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{1}{6}$

   By testing these possible zeros, we find $x = 3$ works, i.e., $p(3) = 0$.

   $\therefore p(x) = 6x^3 - 11x^2 - 19x - 6 = (x-3)(6x^2 + 7x + 2) = (x-3)()6(x + \frac{1}{2})(x + \frac{2}{3})$

   $\therefore$ all real zeros of the polynomial are $-\frac{2}{3}$, $-\frac{1}{2}$ and 3

3. (10%) The demand and supply equations for a certain type of electronic organizer are given by

$$\begin{aligned} p &= 180 - 4x, \\ p &= 75 + 3x, \end{aligned}$$

   where p is the price in dollars and x represents the number of units in thousands. Find the equilibrium point for this market.

   Solution:

$$\begin{aligned} \because\quad & 180 - 4x = 75 + 3x \\ \Leftrightarrow\quad & 105 = 7x \\ \Leftrightarrow\quad & x = 15 \end{aligned}$$

   $\therefore p = 180 - 4 \times 15 = 120$

   $\therefore$ the equilibrium point for this market is $(x, p) = (15, 120)$

4. (10%) You are setting up a part-time business with an initial investment of \$5000. The unit cost of the product is \$11.8, and the selling price is \$19.3. Find equations for the total cost C and total revenue R for x units, and find the break-even point.

Solution:

(a). $C = 5000 + 11.8x$ and $R = 19.3x$

(b). Let $C = R$. Then

$$\begin{aligned} \because\ & 5000 + 11.8x = 19.3x \\ \Leftrightarrow\ & 5000 = 7.5x \\ \Leftrightarrow\ & x = \frac{5000}{7.5} (= 666\frac{2}{3}) \end{aligned}$$

$\therefore$ the break-even point is $(x, C = R) = (\frac{5000}{7.5}, \frac{96500}{7.5})$

5. (10%) Write the "general form" of the equation of the circle given endpoints of a diameter $(3, 3)$ and $(-3, 3)$.

Solution:

$\because$ endpoints of a diameter are $(3, 3)$ and $(-3, 3)$

$\therefore$ center is $(0, 3)$, diameter $= \sqrt{(3 + 3)^2 + 0^2} = 6$ and radius $= 3$

$\therefore$ the equation of the circle is $(x - 0)^2 + (y - 3)^2 = 3^2$

$\therefore$ the general form of the circle is $x^2 + y^2 - 6y = 0$

In Problems 6 $\sim$ 8, find the limit if it exists. If the limit does not exist, explain why.

6. (10%) $\lim_{x\to 1} F(x)$, where $F(x) = \dfrac{x^2 - 1}{|x - 1|}$.

Solution:

$$\begin{aligned} \lim_{x\to 1^+} \frac{x^2 - 1}{|x - 1|} &= \lim_{x\to 1^+} \frac{(x + 1)(x - 1)}{x - 1} = 2 \\ \lim_{x\to 1^-} \frac{x^2 - 1}{|x - 1|} &= \lim_{x\to 1^+} \frac{(x + 1)(x - 1)}{-(x - 1)} = -2 \end{aligned}$$

$\because \lim_{x\to 1^+} F(x) = 2 \neq -2 = \lim_{x\to 1^-} F(x)$

$\therefore$ the limit does not exist

7. (10%) $\lim_{x\to 2} \dfrac{x - 2}{x^2 - 4x + 4}$.

Solution:

$\because \lim_{x\to 2} \dfrac{x-2}{x^2-4x+4} = \lim_{x\to 2} \dfrac{x-2}{(x-2)^2}$ and $\lim_{x\to 2^+} \dfrac{1}{x-2} = \infty$, $\lim_{x\to 2^-} \dfrac{1}{x-2} = -\infty$

$\therefore \lim_{x\to 2} \dfrac{x-2}{x^2-4x+4}$ does not exist

8. (10%) $\lim_{x\to 1/2} (x - 2[x])$, where $[x]$ denotes the greatest integer function.

   Solution:

   $\because \lim_{x\to 1/2} x = \dfrac{1}{2}$ and $\lim_{x\to 1/2} 2[x] = 0$

   $\therefore \lim_{x\to 1/2} (x - 2[x]) = \dfrac{1}{2}$

9. (10%) Find the constants a and b such that the function is continuous on the entire real line:

$$f(x) = \begin{cases} 2, & x \leq -1, \\ ax+b, & -1 < x < 3, \\ -2, & x \geq 3. \end{cases}$$

   Solution:

$$\begin{aligned} \lim_{x\to -1^-} f(x) &= 2 = \lim_{x\to -1^+} ax+b = -a+b, \\ \lim_{x\to 3^+} f(x) &= -2 = \lim_{x\to 3^-} ax+b = 3a+b. \end{aligned}$$

   $\Rightarrow -a+b = 2$ and $3a+b = -2$

   $\therefore a = -1$ and $b = 1$

10. (10%) Discuss the continuity of the function $f(x) = \frac{1}{x-2}$ on the closed interval $[1,4]$. If there are any discontinuities, determine whether they are removable.

    Solution:

    $\because f(x)$ is a rational function

    $\therefore f(x)$ is continuous on its domain

    $\therefore f(x)$ is continuous on $[1,2) \cup (2,4]$ and is discontinuous at $x = 2$

    $\because \lim_{x\to 2^+} \dfrac{1}{x-2} = \infty$ and $\lim_{x\to 2^-} \dfrac{1}{x-2} = -\infty$

    $\therefore x = 2$ is nonremovable