FUNDAMENTAL THEOREM OF CALCULUS

1. State "Fundamental Theorem of Calculus".

Sol. $\left.\begin{array}{l} f \text{ is a continuous function} \\ F(x) = \int_a^x f(t)\,dt \end{array}\right\} \Rightarrow F'(x) = f(x)$

2. (1) Suppose that f is a continuous function and g is a differentiable function.
   Let
   $$H(x) = \int_a^{g(x)} f(t)dt.$$
   Find $H'(x)$.

   (2) Suppose that f is a continuous function and g_1, g_2 are differentiable functions.
   Let
   $$K(x) = \int_{g_1(x)}^{g_2(x)} f(t)dt.$$
   Find $K'(x)$.

Sol. (1) Let $F(x) = \int_a^x f(t)\,dt$.

Then $H(x) = F(g(x))$

$\Rightarrow H'(x) = F'(g(x))\, g'(x)$

$= f(g(x))\, g'(x)$

(2) $K(x) = \int_a^{g_2(x)} f(t)\,dt - \int_a^{g_1(x)} f(t)\,dt$

$\Rightarrow K'(x) = f(g_2(x))\, g_2'(x) - f(g_1(x))\, g_1'(x)$ (by (1))

3. $f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ x^2 & \text{if } 1 < x. \end{cases}$

Let
$$F(x) = \int_{-1}^{x} f(t)dt.$$
Find $F(x)$ and $F'(x)$.

Sol: $x \leq 1$, $\int_{-1}^{x} f(t)dt = \int_{-1}^{x} t+1\, dt$
$= \frac{1}{2}x^2 + x + \frac{1}{2}$

$x > 1$, $\int_{-1}^{x} f(t)dt = \int_{-1}^{1} t+1\, dt + \int_{1}^{x} t^2\, dt$
$= 2 + \frac{1}{3}x^3 - \frac{1}{3}$
$= \frac{1}{3}x^3 + \frac{5}{3}$

$\therefore F(x) = \begin{cases} \frac{1}{2}x^2 + x + \frac{1}{2} & x \leq 1 \\ \frac{1}{3}x^3 + \frac{5}{3} & 1 < x. \end{cases}$

$F'(x) = \begin{cases} x+1 & x < 1 \\ \text{不存在} & x = 1 \\ x^2 & 1 < x \end{cases}$

4. $f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ x^2+1 & \text{if } 1 < x. \end{cases}$

Let
$$F(x) = \int_{-1}^{x} f(t)dt.$$
Find $F(x)$ and $F'(x)$.

Sol. $x \leq 1$, $\int_{1}^{x} f(t)\, dt = \int_{-1}^{x} t+1\, dt$
$= \frac{1}{2}x^2 + x + \frac{1}{2}$

$x > 1$, $\int_{1}^{x} f(t)\, dt = \int_{-1}^{1} t+1\, dt + \int_{1}^{x} t^2+1\, dt$
$= 2 + \frac{1}{3}x^3 + x - \frac{4}{3}$
$= \frac{1}{3}x^3 + x + \frac{2}{3}$

$\therefore F(x) = \begin{cases} \frac{1}{2}x^2 + x + \frac{1}{2} & x \leq 1 \\ \frac{1}{3}x^3 + x + \frac{2}{3} & x > 1 \end{cases}$

$F'(x) = \begin{cases} x+1 & x \leq 1 \\ x^2+1 & 1 < x \end{cases}$