DIFFERENTIATION

Solutions:

1. f is differentiable at a

$$\Rightarrow f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \text{ exists}$$

$$\begin{aligned}\Rightarrow \lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a)\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}(x-a)\\ &= f'(a)\cdot 0\\ &= 0\end{aligned}$$

$$\Rightarrow \lim_{x\to a} f(x) = f(a)$$

$\Rightarrow$ f is continuous at a.

2. (For example) $f(x) = \mid x \mid$

Then f is continuous at 0 , but f is not differentiable at 0

3. f is differentiable at 1

$\Rightarrow$ f is continuous at 1

$$\Rightarrow \lim_{x\to 1^-} ax^2+bx+7 = \lim_{x\to 1^+} 2bx+2a$$

$$\Rightarrow a+x+7 = 2b+2a$$

$$\Rightarrow a+b=7 \qquad \cdots(1)$$

f is differentiable at 1

$$\Rightarrow \begin{cases} f'(1) = \frac{d}{dx}(ax^2+bx+7)\mid_{x=1} \\ \quad\quad = 2ax+b\mid_{x=1} \\ \quad\quad = 2a+b \\ f'(1) = \frac{d}{dx}(2bx+2a)\mid_{x=1} \\ \quad\quad = 2b\mid_{x=1} \\ \quad\quad = 2b \end{cases}$$

$$\Rightarrow 2a+b=2b$$

$$\Rightarrow 2a-b=0 \qquad \cdots(2)$$

$$(1),(2) \Rightarrow \begin{cases} a+b=7 \\ 2a-b=0 \end{cases}$$

$$\Rightarrow a=\frac{7}{3}\ ,\ b=\frac{14}{3}$$

4. (1) $\frac{dy}{dx} = 3\sin^2(\tan x)\cos(\tan x)\sec^2 x$

(2) $\frac{dy}{dx} = \sec(\cos^{\frac{1}{2}}(x^2+5-\frac{1}{x}))\tan(\cos^{\frac{1}{2}}(x^2+5-\frac{1}{x}))\cdot$
$\frac{1}{2}\cos^{-\frac{1}{2}}(x^2+5-\frac{1}{x})\cdot(-\sin(x^2+5-\frac{1}{x}))\cdot(2x+x^{-2})$

(3) $\frac{dy}{dx} = 2x\cot(\csc x) + x^2(-\csc^2(\csc x)(-\csc x\cot x))$
$+\sin x\cos x + x\cos x\cos x + x\sin x(-\sin x)$
$= 2x\cot(\csc x) + x^2\csc^2(\csc x)\csc x\cot x$
$+\sin x\cos x + x\cos^2 x - x\sin^2 x$

(4) $\frac{dy}{dx} = (\cos(x^2+1))\cdot 2x + \frac{(x^3+1)-(x+2)\cdot 3x^2}{(x^3+1)^2}$
$= 2x\cos(x^2+1) + \frac{-2x^3-6x^2+1}{(x^3+1)^2}$

5. $y' = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 3a_3x^2 + 2a_2x + a_1$
$y'' = n(n-1)a_nx^{n-2} + (n-1)(n-2)a_{n-1}x^{x-3} + \cdots + 3\cdot 2a_3x + 2a_2$
$y^{(n)} = a_n \cdot n!$
$y^{(n+1)} = 0$

6. $y^3 = x^2 + \cos\frac{x}{y}$

$\Rightarrow \frac{d}{dx}(y^3) = \frac{d}{dx}(x^2+\cos\frac{x}{y})$

$\Rightarrow 3y^2\frac{dy}{dx} = 2x - (\sin\frac{x}{y})\cdot\frac{y - x\frac{dy}{dx}}{y^2}$

$= 2x - (\sin\frac{x}{y})\cdot\frac{1}{y} + (sin\frac{x}{y})\cdot\frac{x}{y^2}\cdot\frac{dy}{dx}$

$\Rightarrow (3y^2 - \frac{x}{y^2}\sin\frac{x}{y})\frac{dy}{dx} = 2x - \frac{1}{y}\sin\frac{x}{y}$

$\Rightarrow \frac{dy}{dx} = \frac{2x - \frac{1}{y}\sin\frac{x}{y}}{2y^2 - \frac{x}{y^2}\sin\frac{x}{y}}$

7. $xy = \sin x + \cos y$

$\Rightarrow \frac{d}{dx}(xy) = \frac{d}{dx}(\sin x + \cos y)$

$\Rightarrow y + x\frac{dy}{dx} = \cos x - \sin y\frac{dy}{dx}$

$\Rightarrow (x+\sin y)\frac{dy}{dx} = \cos x - y$

$\therefore \frac{dy}{dx} = \frac{\cos x - y}{x + \sin y}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{\cos x - y}{x+\sin y})$

$$= \frac{(-\sin x - \frac{dy}{dx})(x+\sin y) - (1+\cos y\frac{dy}{dx})(\cos x - y)}{(x+\sin y)^2}$$

$$= \frac{-x\sin x - \sin x\sin y - \cos x + y - (x+\sin y+\cos y\cos x - y\cos y)\frac{dy}{dx}}{(x+\sin y)^2}$$

$$= \frac{-x\sin x - \sin x\sin y - \cos x + y - (x+\sin y+\cos y\cos x - y\cos y)\frac{\cos x - y}{x+\sin y}}{(x+\sin y)^2}$$

8.

(1) $\begin{cases} 2^4 - 4\cdot 2^2 = 0 \\ 3^4 - 9\cdot 3^3 = 0 \end{cases}$

$\Rightarrow 2^4 - 4\cdot 2^2 = 3^4 - 9\cdot 3^2$

$\Rightarrow$ (3,2) is on the curve C.

Similarly, for (3,-2).

(2) $y^4 - 4y^2 = x^4 - 9x^2$

$\Rightarrow \frac{d}{dx}(y^4 - 4y^2) = \frac{d}{dx}(x^4 - 9x^2)$

$\Rightarrow 4y^3\frac{dy}{dx} - 8y\frac{dy}{dx} = 4x^3 - 18x$

$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 18x}{4y^3 - 8y}$

$= \begin{cases} \frac{27}{8} & (x,y) = (3,2) \\ -\frac{27}{8} & (x,y) = (3,-2) \end{cases}$

$\therefore$ *Equation* : $y - 2 = \frac{27}{8}(x-3)$

(3) slope of the tangent $\cdot$ slope of the normal = -1

$\Rightarrow -\frac{27}{8}\ \cdot$ slope of the normal = -1

$\Rightarrow$ slope of the normal = $\frac{8}{27}$

$\therefore$ *Equation* : $y + 2 = \frac{8}{27}(x-3)$

9. (1) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dy}{dt}}$

$= \frac{\sin t}{1-\cos t}$

$\frac{dy}{dx}\Big|_{t=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}}$

$$=\frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}}$$
$$=\sqrt{3}$$

$$\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx}$$
$$=\frac{dy'}{dx}$$
$$=\frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$
$$=\frac{\frac{\cos t(1-\cos t)-\sin t\cdot\sin t}{(1-\cos t)^2}}{1-\cos t}$$
$$=\frac{\cos t-1}{(1-\cos t)^2}\cdot\frac{1}{1-\cos t}$$
$$= - \frac{1}{(1-\cos t)^2}$$

$$\frac{d^2y}{dx^2}\,|_{t=\frac{\pi}{3}} = -\frac{1}{(1-\cos t)^2}$$
$$=-\frac{1}{4}$$

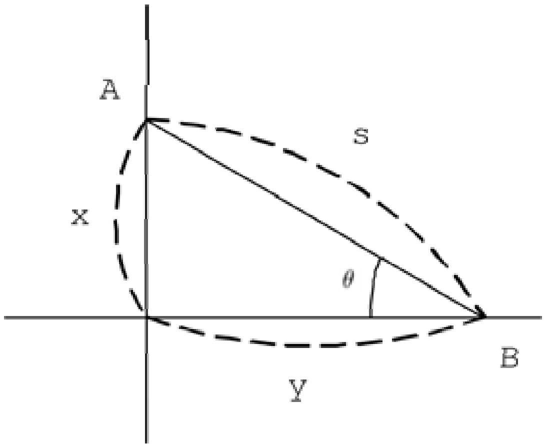
(2) t= $\frac{\pi}{3}$

$x = \frac{\pi}{3} - \sin\frac{\pi}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$

$y = 1 - \cos\frac{\pi}{3} = \frac{1}{2}$

Equation : $y - \frac{1}{2} = \sqrt{3}(x - \frac{\pi}{3} + \frac{\sqrt{3}}{2})$

10. Let x = distance between A and the intersection
y = distance between B and the intersection
s = distance between A and B



Then $\begin{cases} \tan\theta = \dfrac{x}{y} \\ s^2 = x^2 + y^2 \end{cases}$

Now $\dfrac{dx}{dt}$ = - 3 m / sec , $\dfrac{dy}{dt}$ = 2 m / sec

$$\tan\theta = \frac{x}{y}$$

$$\Rightarrow \frac{d}{dt}\tan\theta = \frac{d}{dt}(\frac{x}{y})$$

$$\Rightarrow \sec^2\theta\frac{d\theta}{dt} = \frac{\frac{dx}{dt}\cdot y - x\frac{dy}{dt}}{y^2}$$

x = 10 m , y = 25 m

$$\Rightarrow \tan\theta = \frac{10}{25} = \frac{2}{5}$$

$$\Rightarrow \sec^2\theta = \frac{29}{25}$$

$$\therefore \frac{29}{25}\frac{d\theta}{dt} = \frac{-3\cdot 25 - 10\cdot 2}{25^2}$$

$$\frac{d\theta}{dt} = -\frac{19}{145} \quad (\text{ rad / sec }) \qquad \cdots(1)$$

$$s^2 = x^2 + y^2$$

$$\Rightarrow \frac{d}{dt}s^2 = \frac{d}{dt}(x^2 + y^2)$$

$$\Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

x = 10 m , y = 25 m

$$\Rightarrow s = \sqrt{10^2 + 25^2} \text{ m}$$
$$= 5\sqrt{29} \text{ m}$$

$$\therefore\ 2 \cdot 5\sqrt{29} \cdot \frac{ds}{dt} = 2 \cdot 10 \cdot (-3) + 2 \cdot 25 \cdot 2$$

$$\frac{ds}{dt} = \frac{4}{\sqrt{29}} \quad (\text{ m / sec }) \qquad \cdots (2)$$

11. (1) $f(x) = x^{\frac{1}{3}}$

$$\Rightarrow f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$\Rightarrow df = f'(x)dx$$

$$= \frac{1}{3}x^{-\frac{2}{3}}dx$$

(2) Let $f(x) = \sqrt[3]{x}$

Note $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$ (Δ x is small)

Take x = 1000 , $\Delta x = 0.2$

$$f(1000.2) \approx f(1000) + f'(1000) \cdot 0.2$$

$$= 10 + \frac{1}{3} \cdot \frac{1}{100} \cdot 0.2$$

$$\approx 10.0006$$

$$\therefore\ \sqrt[3]{1000.2} \approx 10.0006$$