LIMIT AND CONTINUITY

Solutions:

1. For every $\epsilon > 0$, there exists $\delta > 0$ such that

   $\mid f(x) - l \mid < \epsilon$ for all x with $0 < \mid x - a \mid < \delta$.

   (or For every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for all x , $0 < \mid x - a \mid < \delta \Rightarrow \mid f(x) - l \mid < \epsilon$.)

2. Arbitrarily give $\epsilon > 0$.

   $\lim_{x \to a} f(x) = l$
   $\Rightarrow$ there exists $\delta_1 > 0$ such that
   $(0 < \mid x - a \mid < \delta_1 \Rightarrow \mid f(x) - l \mid < \frac{\epsilon}{2})$ $\quad \ldots (1)$

   $\lim_{x \to a} g(x) = m$
   $\Rightarrow$ there exists $\delta_2 > 0$ such that
   $(0 < \mid x - a \mid < \delta_2 \Rightarrow \mid g(x) - m \mid < \frac{\epsilon}{2})$ $\quad \ldots (2)$

   Take $\delta = \min\{\delta_1, \delta_2\}$, then

   $0 < \mid x - a \mid < \delta$

   $$\Rightarrow \begin{cases} 0 < \mid x - a \mid < \delta_1 \\ 0 < \mid x - a \mid < \delta_2 \end{cases}$$

   $$\Rightarrow \begin{cases} \mid f(x) - l \mid < \frac{\epsilon}{2} & (\because (1)) \\ \mid g(x) - m \mid < \frac{\epsilon}{2} & (\because (2)) \end{cases}$$

   $\Rightarrow \mid (f(x) + g(x) - (l + m) \mid$

   $= \mid (f(x) - l) + (g(x) - m) \mid$

   $\leq \mid f(x) - l \mid + \mid g(x) - m \mid$

   $< \epsilon$

$\therefore$ For an arbitrarily given $\epsilon > 0$, there exist $\delta > 0$ such that

$$0 <\mid x - a \mid< \delta_1 \Rightarrow\mid f(x) + g(x) - (l+m) \mid< \epsilon.$$

Hence $\lim_{x \to a} f(x) + g(x) = l + m.$

3. Suppose $g(x) \leq f(x) \leq h(x)$ for all x in some neighborhood of a , and $x \neq a$.
   If $\lim_{x \to a} g(x) = l$ and $\lim_{x \to a} h(x) = l$, then $\lim_{x \to a} f(x) = l$.
4. $\lim_{x \to a} f(x)$ exists, and $\lim_{x \to a} f(x) = f(a)$
5. (1) 3 (2) $\frac{1}{2}$ (3) does not exist
6. (i) (1)1 (2) 2 (3) does not exists
   (ii) g is not continuous at 0.
   $\because \lim_{x \to 0} g(x)$ does not exist.
7. (i) (1) 2 (2) 2 (3) 2
   (ii) h is continuous at -1.
   $\because \lim_{x \to -1} h(x) = h(-1)$
8. (i) (1) 0 (2) does not exist
   (ii) 0
9. $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

   $\because$ When x approaches 0 , the values of $\sin \frac{1}{x}$ change between -1 and 1 ; they do not approach a fixed number.
10. $- \mid x \mid\leq x \sin(\frac{1}{x}) \leq\mid x \mid$ for all x $\neq 0$, and

$$\begin{cases} \lim_{x \to o} \mid x \mid= 0 \\ \lim_{x \to o} - \mid x \mid= 0 \end{cases}$$

Hence , by Sandwich Theorem , $\lim_{x \to o} x \sin \frac{1}{x} = 0$

11. For every M > 0 , there exists $\delta > 0$ such that

$$0 <\mid x - a \mid< \delta \Rightarrow f(x) > M.$$